THE FOUNDATIONS OF A THEORY OF THE CALCULUS OF VARIATIONS IN THE LARGE IN m-SPACE (SECOND PAPER)*

BY

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1. Introduction. The general problem† is the classification and existence of extremals under all kinds of conditions, a non-linear boundary value problem in the large. This paper deals primarily with the fixed end point problem.

The first paper by the author dealt with the fixed end point problem in the small, and with the problem with one end point variable in general. The latter problem could be readily studied in the large because it corresponds to the problem of finding the critical points of a function whose critical points are in general isolated.

Such is not the case in the fixed end point problem. The critical sets appearing there take the form of n-dimensional loci. This extreme difficulty is met by the aid of deformations. The author's characterization of the number of conjugate points on an extremal by means of deformations is essential. See Morse II.

A second difficulty which long appeared insurmountable was the fact

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† The following references should be made.

For work on the absolute minimum including work of Hilbert see Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 419–437, and Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2. Further references will be found in these works.

Signorini, Rendiconti del Circolo Matematico di Palermo, vol. 33 (1912), pp. 187–193. Here broken geodesics are used in the quest for minimizing periodic orbits.

Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9. In chapter V, Birkhoff makes new and effective use of deformations of broken geodesics to obtain basic periodic motions. His coupling of the minimum and minimax methods was undoubtedly the first step towards a general theory.

The following papers by the author will be referred to: I These Transactions, vol. 27 (1925), pp. 345–396. II These Transactions, vol. 30 (1928), pp. 213–274. III These Transactions, vol. 31 (1929), pp. 379–404. IV Mathematische Annalen, vol. 103 (1930), pp. 52–69.

See also the following abstract: MM. Lusternik et Schnirelmann, *Existence de trois géodésiques fermées sur toute surface de genre* 0, Comptes Rendus, vol. 188 (1929), No. 8, p. 534.

For references to Bliss, Mason, Carathéodory, and Hadamard, see the papers Morse II and III.

that the domains of definition of the functions whose critical points were sought were not complexes, or at least could not readily be reduced to finite complexes.

This difficulty was again surmounted with the aid of deformations. The regions involved were shown to be deformable on themselves into subregions which were complexes. The existence of these deformations was a matter of the calculus of variations, their use a matter of analysis situs. Both aspects were indispensable.

A final difficulty as well as source of great interest was the fact that the domains usually have infinite connectivities. From the point of view of the calculus of variations this involved a study of the density of infinite sets of conjugate points of a fixed point. Over against the infinite set of connectivities is set a conjugate number sequence belonging to the calculus of variations, and the interrelations of these two infinite sequences appear central.

As a particular example, one can prove the existence of infinitely many geodesics joining any two points on any regular topological image of an m-sphere.

The author wishes to state in passing that a considerable part of the analysis of this problem has been carried over to the problem of periodic extremals.* The latter problem, however, presents additional difficulties both from the point of view of the calculus of variations and that of analysis situs. It involves the study of homologies on a given complex among complexes which are restricted to product complexes. The simplest complexes present decidedly new aspects from this point of view.

I. THE PROBLEM AND ITS DEFORMATIONS

2. The hypotheses. Let S be a closed region representable as an m-dimensional complex† A_m in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x, r) \tag{2.1}$$

be a *positive, analytic* function of its arguments for (x) on a region slightly larger than S, and for (r) any set not (0). Suppose further that F is positively homogeneous of order one in the variables (r). We employ the parametric form, taking $F(x, \dot{x})$ as our integrand and

$$J = \int_{t_1}^{t_2} F(x, \dot{x})dt$$

* See Marston Morse, *Closed extremals*, Proceedings of the National Academy of Sciences, vol. 15 (1929), p. 856.

† See Veblen, The Cambridge Colloquium, 1916, Part II, *Analysis Situs*. Terms in analysis situs will be used in the sense of Veblen unless otherwise defined.

as our integral, where $(\dot{x})$ stands for the set of derivatives of (x) with respect to the parameter t. We suppose that the problem is ***positively regular***, that is, that

$$\sum_{ij} F_{r_i r_j}(x, r)\eta_i\eta_j > 0 \qquad (i, j = 1, 2, \cdots, m) \tag{2.2}$$

for (x) and (r) as before, and (η) any set not (0) nor proportional to (r).

*We suppose that S's boundary is extremal-convex.**

That is, we suppose there exists a positive constant e so small that any extremal segment on which $J<e$ and which joins two boundary points of S will lie interior to S except at most for its end points.

Let P and Q be any two distinct points on S. Let there be given an extremal segment g which joins P to Q on S. Let λ be the direction of g at P. Let the direction cosines of the directions neighboring λ be regularly† represented as functions of $m-1$ parameters (α). Let s be the value of J taken along the extremal through P in the direction determined by (α). On the extremals issuing from P with the directions determined by (α) the coordinates (x) will be analytic functions of s and (α).

All of the extremal segments through P neighboring g cannot go through Q.

This could happen if S were a closed manifold, but it cannot happen if S is an extremal-convex region. For if all such extremals did go through Q one could prove first that s would be a constant at Q, and secondly, one could show by a process of analytic continuation that all extremals whatsoever through P would reach Q at least as soon as they reach the boundary.

This cannot occur. For in particular the class of extremals which join P to an arbitrary point $R \neq Q$ of the boundary, and give an absolute minimum to J between P and R, cannot all pass through Q prior to R without violating their minimizing property. Thus the statement in italics is proved.

If in terms of the parameters s and (α) one writes down the m conditions that the extremal through P pass through Q, one sees from the theory of analytic functions that the extremals which join P to Q on S with J less than a finite constant J_0, are conditioned as follows:

These extremals are either finite in number or else make up a finite set of families on which (x) is representable by means of the parameters s and (α),

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* Instead of assuming that S was an extremal-convex region in m-space, we could have assumed that S was a regular $(m-1)$-dimensional manifold in m-space. Such a change would entail at most obvious changes in the following. The results for this case will be reviewed at the end of the paper.

† That is, the direction cosines are to be analytic functions of the parameters (α) of such sort that not all of the jacobians of $m-1$ of the direction cosines with respect to the parameter (α) are zero.

with s a constant for each family, and (α) *analytic in general for each family on a suitably chosen "Gebilde"* of r independent variables with* $0<r<m-1$.

We note the following.

In the plane there are at most a finite number of extremals on S through P and Q on which $J<J_0$.

We wish to call attention to the case where Q is not conjugate to P on any of the extremals joining P to Q on which $J<J_0$. This is the general case. In this case there are at most a finite number of extremals joining P to Q on which $J<J_0$, as is readily seen from the fact that each such extremal is isolated.

3. The region Σ and function $J(\pi)$. Instead of considering the set of all curves joining P to Q our purposes will be equally well served by considering a particular class of broken extremals joining P to Q. To proceed we shall give successively a number of definitions.

We shall call the value of J taken along a curve γ the *J-length* of γ.

The constant J_0. We shall restrict ourselves to curves of J-length less than J_0, where J_0 is a positive constant larger than the absolute minimum of J along curves joining P to Q, and where further J_0 is not equal to the value of J along any extremals joining P to Q.

The constant ρ. It is well known that there exists a positive constant e_1 small enough to have the following properties. Any extremal E lying on S and with a J-length less than e_1, will give an absolute minimum to J relative to all curves of class D' joining its end points. The coördinates (x) of any point on E will be analytic functions of the coördinates of the end points of E and of the distance of (x) from the initial end point O of E, at least as long as E does not reduce to a point. The set of all extremal segments issuing from O with J-lengths equal to e_1 will form a field covering a neighborhood of O in a one-to-one manner, O alone excepted. With e_1 thus chosen we now choose the positive constant ρ so that

$$\rho < e_1, \quad \rho < e, \tag{3.1}$$

where e is the constant used in the definition of the term extremal-convex.

Any extremal segment on S whose J-length is at most ρ will be called an *elementary extremal.*

Admissible broken extremals. Let n be a positive integer so large that

$$J_0 < (n+1)\rho. \tag{3.2}$$

Consider now the class of broken extremals g which satisfy the following conditions:

* See Osgood, *Funktionentheorie*, II.

I. The broken extremal joins P to Q on S.

II. It consists of $n+1$ elementary extremals.

III. Its J-length is less than J_0.

Such a broken extremal g will be termed *admissible.*

Admissible points (π). Let

$$P, P_1, \cdots, P_n, Q \tag{3.3}$$

be the successive ends of the elementary extremals of g. We admit the possibility that two or more of these points be coincident. Let (π) represent the set of mn variables which give the coördinates of the points

$$P_1, P_2, \cdots, P_n, \tag{3.4}$$

taking first the coördinates of P_1, then those of P_2, and so on. The points (3.4) will be called the vertices of g. A point (π) derived from the vertices of an admissible broken extremal will itself be called *admissible.*

There are infinitely many admissible points (π). In particular let g_1 be an extremal joining P to Q with a J-length less than J_0. If we divide g_1 into $n+1$ successive extremals of equal J-length, these extremals turn out to be elementary extremals by virtue of (3.2), so that the resulting n points of division of g_1 combine into an admissible point (π). This point (π) has the further property that any point (π) in its neighborhood will also be admissible.

The set of all admissible points (π) will form an mn-dimensional domain Σ.

The value of the integral J taken along the broken extremal determined by an admissible point (π) from the point P to the point Q will be denoted by $J(\pi)$.

4. The boundary of Σ. Corresponding to an admissible point (π), let $M(\pi)$ be the maximum of the J-lengths of the elementary extremals that make up the broken extremal determined by (π).

The boundary of Σ will consist of points (π) of one or more of the following types:

Type I: Points at which $J(\pi)=J_0$;

Type II: Points at which $M(\pi)=\rho$;

Type III: Points corresponding to which at least one vertex P_i lies on the boundary of S.

The function $J(\pi)$ is analytic on Σ at least as long as successive vertices remain distinct. A point (π) corresponding to which the successive vertices are distinct will be called a *critical point* of $J(\pi)$ if all of the first partial derivatives of $J(\pi)$ are zero at that point. We shall prove for the case of distinct vertices that $J(\pi)$ will have a critical point (π) when and only when the corresponding broken extremal reduces to an unbroken extremal γ joining P to Q.

For the partial derivative of $J(\pi)$ with respect to the ith coördinate of a vertex (x), in the case of distinct vertices, is seen to be

$$F_{r_i}(x, p) - F_{r_i}(x, q) \qquad (i = 1, 2, \cdots, m) \tag{4.1}$$

where (p) and (q) give the directions at (x) of the elementary extremals preceding and following (x) respectively. If the differences (4.1) all vanished we would have

$$\sum_i [p_i F_{r_i}(x, p) - p_i F_{p_i}(x, q)] = 0. \tag{4.2}$$

But the sum (4.2) equals the Weierstrass E-function $E(x, p, q)$ which is known not to vanish* for $(p) \neq (q)$ whenever the hypothesis of regularity of §2 is granted. Hence $(p) = (q)$ if $J(\pi)$ has a critical point. The corresponding extremal can therefore have no corners.

If account be taken of the freedom in moving vertices on an extremal γ joining P to Q, it is seen that each such extremal corresponds to an n-dimensional set of critical points (π).

5. **The deformation T.** The boundary of Σ is exceedingly complex. We can avoid examining its structure more closely by showing how to deform Σ continuously as a whole into a part of Σ which contains none of the boundary points of Σ of types I and II. These deformations amount to deformations of broken extremals. For future use they need to have the following properties.

(α) *They do not increase $J(\pi)$ or $M(\pi)$ beyond their initial values.*

(β) *They deform admissible broken extremals into admissible broken extremals.*

(γ) *They deform continuous families of broken extremals continuously.*

Such deformations will be called J-deformations.

The deformation D_1. On an admissible broken extremal g let a point U be given. Let the value of J taken along g from P to U be termed the *J-coördinate, u,* of the point U on g. In case u is a function of the time t it will be convenient to term du/dt the *J-rate* of U on g.

As the time t varies from 0 to 1 let the n vertices P_i on g move along g from their initial positions to a set of positions which divide g into $n+1$ successive segments of equal J-lengths, each vertex moving at a constant J-rate.

We term this the deformation D_1. It is readily seen that D_1 is a J-deformation. We also note the following.

* See Bliss, *The Weierstrass E-function for problems of the calculus of variations in space*, these Transactions, vol. 15 (1914), pp. 369–378.

LEMMA 5.1. *The deformation D_1 carries each admissible broken extremal into one for which*

$$M(\pi)(n+1) < J_0.$$

The deformation D_2. Let h_i be the ith elementary extremal of g. As the time t increases from 0 to 1 let points $H_i(t)$ and $K_{i+1}(t)$ start from P_i and move away from P_i, respectively on h_i and h_{i+1}, at J-rates equal to one half the J-lengths of h_i and h_{i+1}. Let $H_i(t)$ and $K_{i+1}(t)$ be joined by an elementary extremal $E_i(t)$. The deformation D_2 is hereby defined as one in which P_i is replaced for each t by the point $P_i(t)$ which divides the elementary extremal $E_i(t)$ into two segments of equal J-lengths.

To show that D_2 is a J-deformation we shall first show that $M(\pi)$ is not increased beyond its initial value.

To that end let h_i also represent the J-length of h_i. Let $g(t)$ be the broken extremal replacing g at the time t. Note that the end points of the ith elementary extremal of $g(t)$ are also connected by a set of three elementary extremals joining successively the four points

$$P_{i-1}(t), \quad K_i(t), \quad H_i(t), \quad P_i(t), \tag{5.1}$$

the values $i=1$ and $n+1$ excepted. The J-lengths of these three elementary arcs are respectively at most

$$\frac{t}{4}(h_{i-1}+h_i), \quad (h_i - th_i), \quad \frac{t}{4}(h_i+h_{i+1}). \tag{5.2}$$

If the three constant h_{i-1}, h_i, and h_{i+1} be replaced by their maximum, the sum of the three quantities in (5.2) is readily seen to be at most that maximum. Thus the deformation D_2 does not increase $M(\pi)$ beyond its initial value.

With this established one sees readily that the remaining properties of J-deformations as previously listed are satisfied by D_2.

The deformations D_1 and D_2 combine to give us a deformation T concerning which we have the following lemma.

LEMMA 5.2. *If σ be a closed set of admissible points (π) at a positive distance from the set of critical points of $J(\pi)$, the product deformation $T=D_1D_2$ will carry each point (π) of σ into an admissible point (π') for which*

$$J(\pi') < J(\pi) - d,$$

where d is a positive constant.

In the region Σ let N be a closed neighborhood of the critical points of $J(\pi)$ so small that N and σ have no points in common. We divide the points (π) of Σ into three classes according to the nature of their images $(\pi)_1$ under D_1 at the time $t=1$.

Class I shall contain (π) if $(\pi)_1$ determines a broken extremal with at least one elementary extremal k of zero J-length.

Class II shall contain (π) if $(\pi)_1$ is in N and (π) is not in Class I.

Class III shall contain those points of σ which are neither in Class I nor in Class II.

The proof of the lemma will be given separately for each class.

Class I. To define D_1 we divided the given broken extremal into segments of equal J-length. When (π) is in Class I, one of these segments, say h, is replaced under D_1 by the elementary extremal k of zero J-length. Since the J-length of h is not zero, $J(\pi)$ must be lessened under D_1.

Class II. In general the only points (π) for which $J(\pi)$ is not lessened under D_1 are those for which the broken extremal determined by (π) is identical with the one determined by $(\pi)_1$. Hence $J(\pi)$ is lessened by D_1 for every point (π) of Class II.

Class III. Here the broken extremal determined by $(\pi)_1$ has no elementary extremals of zero J-length, and has at least two successive elementary extremals which intersect at an angle which is not straight. The application of D_2 to $(\pi)_1$ will then lessen $J(\pi)_1$.

Thus under $T=D_1D_2$, $J(\pi)$ will be lessened for each point of σ. Since σ is closed, and T is continuous on σ, $J(\pi)$ will be lessened on σ by at least some definite positive constant d, independent of (π) on σ. Thus the lemma is proved.

6. **The domain (a, ρ) and its connectivities.** Let a be a non-critical value of $J(\pi)$, and r any positive constant at most the constant ρ of §3.

The set of points (π) which satisfy

$$J(\pi) < a,\ M(\pi) < r,\ a < J_0 \tag{6.1}$$

will be called a domain (a, r).

We shall be more particularly concerned with domains (a, r) for which $r=\rho$. Because of the complicated nature of the boundary of a domain (a, ρ) it is not feasible to try to break such a domain up into cells. Because of our J-deformations it is fortunately possible to study the topological properties of the domain (a, ρ) by means of approximating complexes.

By hypothesis the region S of the space of the m variables (x) can be broken up into m-cells so as to form an m-complex A_m. We provide n copies of A_m, namely

$$A_m^1,\ A_m^2,\ \cdots,\ A_m^n, \tag{6.2}$$

upon which, in particular, we suppose the n vertices, respectively, of an admissible point (π) lie. The product complex A_{mn} formed by taking an arbitrary point from each of the complexes (6.2) will represent a portion of the space of the points (π) of which each domain (a, r) will be a subset.

By virtue of our choice of the integer n in (3.2) we can choose a positive constant e so small that

$$J_0 < (\rho - e)(n + 1), \tag{6.3}$$

choosing e also so small that no constant on the interval between a and $a-e$, including $a-e$, is a critical value of $J(\pi)$. If the deformation T be applied to the domain (a, ρ) it follows from (6.3) and Lemma 5.1 that the resulting points (π) will be such that

$$M(\pi) < \rho - e.$$

It follows then from Lemma 5.2 that a sufficient number of iterations of T will J-deform the domain (a, ρ) on itself into a set of points on the domain $(a-e, \rho-e)$.

Let C_{mn} be a sub-complex of cells of A_{mn} that contains all of the points of $(a-e, \rho-e)$. If A_{mn} be sufficiently finely divided, C_{mn} may be chosen so as to consist only of points on (a, ρ). It will serve as our e-approximation of (a, ρ).

As we have seen, any complex of A_{mn} on (a, ρ) can be J-deformed on (a, ρ) into a complex on C_{mn}. It follows that any cycle* on (a, ρ) is homologous (always mod 2) on (a, ρ) to a cycle of cells of C_{mn}. Therefore all cycles on (a, ρ) are homologous to a finite number of such cycles.

By a complete *j-set of non-bounding* j-cycles on (a, ρ) will be meant a set of j-cycles $(K)_j$ on (a, ρ) with the following properties.

I. Every j-cycle on (a, ρ) is homologous to a linear combination of members of $(K)_j$.

II. There are no proper† homologies on (a, ρ) between the members of $(K)_j$.

The following is an almost immediate consequence of the definition.

The number of j-cycles in a complete j-set for (a, ρ) *is the same for all such complete j-sets.*

The jth connectivity number R_j *of the domain* (a, ρ) *will now be defined as the number of j-cycles in a complete j-set.*

We come now to a first theorem.

THEOREM 1. *If a and b,* $a<b$, *are two non-critical values of* $J(\pi)$ *between which there are no critical values of* $J(\pi)$, *the connectivities of the domains* $(a, \rho)=A$ *and* $(b, \rho)=B$ *are the same.*

* By an i-cycle $(i>0)$ is meant a set of i-circuits. A 0-cycle shall mean any finite set of points.

† That is, homologies which are not empty (mod 2).

It will be sufficient to prove that a complete j-set for A, say $(K)_j$, is a complete j-set for B.

That every j-cycle on B is homologous to a linear combination of j-cycles of $(K)_j$ follows from the fact that the domain B can be J-deformed on itself into a set of points on A. It remains to prove that there are no homologies on B between the members of $(K)_j$.

If now a j-cycle, say H_j, composed of j-cycles of $(K)_j$ bounded a complex H_{j+1} on B, it would follow from Lemma 5.2 that we could J-deform H_{j+1} on B into a complex H'_{j+1} on A. The boundary of H'_{j+1} would be homologous on A to H_j, so that H_j would be bounding on A contrary to the choice of $(K)_j$.

Thus $(K)_j$ is a complete j-set for B. The theorem follows at once.

7. The preliminary homologies. Let a and b be two non-critical values of $J(\pi)$ between which $J(\pi)$ takes on just one critical value $J(\pi)=c$. Let us suppose that this critical value corresponds to a single extremal g, of type k, joining P to Q. We are going to show that the J-connectivities of the domains $A=(a, \rho)$ and $B=(b, \rho)$ differ only when $j=k$ or $k-1$.

Let ω denote the set of critical points of $J(\pi)$ on B at which $J(\pi)$ takes on the value $J(\pi)=c$.

Each point (π) of ω will determine the above extremal g. We come now to the following lemma.

LEMMA 7.1. *If there be given on B an arbitrary neighborhood R of the critical points ω, then in any sub-neighborhood R' sufficiently small every j-circuit is homologous to zero on R.*

It will be sufficient to show that, when suitably chosen, R' can be continuously deformed on R into a point on R. We shall prove that the required deformation can be effected by the product of three deformations $D_1D_3D_1$ of which D_3 will now be defined.

The deformation D_3. Let the extremal g be slightly extended at its ends to form an extremal g'. From each of the n vertices of points (π) defining a broken extremal near g we drop a perpendicular on S to g'. We let the vertices of (π) move along these perpendiculars towards g' with velocities equal to the lengths of the perpendiculars. As the time t varies from 0 to 1 the points (π) with vertices near g' will be deformed into points (π) with vertices on g'. This is the deformation D_3.

Observe now that the deformation D_1 will carry each point of ω into that point $(\pi)_0$ of ω whose vertices divide g into $n+1$ segments of equal J-length. Let N be a neighborhood of the point $(\pi)_0$ so small that the deformation D_3 will carry points (π) on N into points (π) defining g, rather than g multiply covered in part.

If now R' be sufficiently small, D_1, being continuous relative to different points (π), will deform R' on R into points near $(\pi)_0$, or in particular into N. Under D_3 the points (π) on N will be deformed on R into points (π) defining g, while the final application of D_1 will carry the resultant points (π) into the point $(\pi)_0$. Thus $D_1D_3D_1$ will serve as the required deformation.

In terms of the critical value c we have the following lemma.

LEMMA 7.2. *If R' be any neighborhood on $B-A$ of the points of the critical set ω, then any j-circuit K_j on B satisfies an homology,*

$$K_j \sim C_j^1 + C_j^2, \text{ on } B,$$

where C_j^1 is a complex on R' and C_j^2 satisfies $J(\pi) < c - e_1$, where e_1 is a positive constant which depends only on B.

To prove the theorem we shall first show that a sufficiently large number of iterations of the deformation T of Lemma 5.2 will carry each point of B either into R', or else into a point at which $J(\pi)$ is less than c.

Note first that T carries each point of ω into a point of ω and recall that T is continuous. Let R be a sub-neighborhood of R' so small that T will carry no point of R outside of R'.

The set of points $B-A-R$ has a positive distance from the critical points of $J(\pi)$, and according to Lemma 5.2 will then be carried by T into points (π') at which

$$J(\pi') < J(\pi) - d, \qquad d > 0,$$

where it will be convenient to suppose $d < c - a$.

Now a sufficiently large number of iterations of T, say T^r, will carry B into a set of points (π) which satisfy

$$J(\pi) < c + \frac{d}{2}\cdot$$

It follows that T^rT will carry all points of B whose rth images are not on R into points at which

$$J(\pi) < c + \frac{d}{2} - d = c - \frac{d}{2} > a, \tag{7.1}$$

while the remaining points of B will be carried into R'.

Let K_j^1 be the image of K_j under T^rT. We now take C_j^2 as the sum of the j-cells on K_j^1 which satisfy (7.1) at some one point at least, and take for C_j^1 the sum of the remaining j-cells of K_j^1. Thus C_j^1 will lie on R'. If K_j^1 be sufficiently finely divided it appears from (7.1) that on C_j^2, $J(\pi)$ will be less than $c - d/4$. Thus the lemma is proved.

We state the obvious extension.

LEMMA 7.3. *If R' be a neighborhood on $B-A$ of the points ω, then any complex K_i on B whose boundary is on A, is homologous on B to a complex $C_i^1+C_i^2$, which has the same boundary as K_i, and where C_i^1 and C_i^2 are complexes as in Lemma 7.2.*

8. **Deformations neighboring critical points.** Let us return to the space of the m variables (x). In an earlier paper we have proved for the plane essentially the following. (See Theorem 9 Morse II.*)

LEMMA 8.1. *On the extremal g let there be k points conjugate to P, with $k>0$, and Q not conjugate to A.*

Then corresponding to any sufficiently small neighborhood N of g, there exists within N an arbitrarily small neighborhood N' of g, such that closed m-families of curves which join g's end points on N', and satisfy

$$J \leqq c - e, \tag{8.1}$$

where e is a sufficiently small positive constant, are conditioned as follows:

(a) *Those for which $m \neq k-1$ can be deformed on N, without increasing J, into a family of lower dimensionality.*

(b) *Those for which $m=k-1$ include a $(k-1)$-family Z_{k-1} composed of admissible broken extremals, which is non-bounding† on N and* (8.1), *and which is such that every other $(k-1)$-family can be deformed on N without increasing J, either into Z_{k-1} or else into a single curve.*

The proof of this theorem in the plane depended in the first instance upon Theorem 2 of Morse II. The later theorem has been generalized for m-space and appears as Theorem 2 of Morse III.

We note the following differences between the present theorem and Theorem 9 of Morse II.

The deformations here are affirmed not to increase J, instead of simply satisfying (8.1). The deformations used in Morse II are products of the deformation D' of §29, Morse II, which as defined does not increase J, and of the deformation (3) of §31, Morse II. Now the latter deformation does increase J, but if in §31, (2) and (3), we replace e^2 by a^2, this deformation does not increase J, while the remainder of §31 holds as before. The remaining deformations of §31 are on S_{k-1} of Morse II, and hence will not increase J.

* As far as the developments of the paper Morse II are concerned, and also for the present paper, we need not restrict the nature of m-families by requiring that the domain of the parametric point be a manifold, but simply that it be a complex, and this extension we suppose made.

† That is, whose "parametric complex" is non-bounding relative to other admissible parametric complexes.

Returning to the space of the points (π) we have the following theorem.

THEOREM 2. *On the extremal g suppose there are k points conjugate to P, with $k>0$, and P not conjugate to Q.*

Then corresponding to any sufficiently small neighborhood R of the critical set ω, there exists within R an arbitrarily small neighborhood R' of ω, such that the circuits K_m on R' which satisfy

$$J(\pi) \leqq c - e, \tag{8.2}$$

where e is a sufficiently small positive constant, are conditioned as follows:

(a) *Those for which $m \neq k-1$ are homologous to zero on R and* (8.2).

(b) *Thcse for which $m=k-1$ include a $(k-1)$-circuit D_{k-1} that is not homologous to zero on R and* (8.2).

(c) *Those for which $m=k-1$ are either homologous to zero or to D_{k-1}, on R and* (8.2).

This theorem will follow from Theorem 1 once we have examined the relations between neighborhoods R of the set ω in the space (π) and neighborhoods N of g in the space (x).

Corresponding to a variable neighborhood R of the set ω there exists a variable neighborhood N of g that contains all the broken extremals determined by points (π) on R and which approaches g as R approaches ω.

Corresponding to a variable neighborhood N of g there exists a variable neighborhood R of ω that contains all the points (π) determined by those broken extremals on N which satisfy $J<c$, and which approaches ω as N approaches g. This follows from the generalization in m-space of (B) §29, Morse II.

With this understood the theorem follows readily.

II. THE ANALYSIS SITUS

9. Homologies among j-complexes when j is not the type number. For future use we now choose on the domain B distinct from A two closed neighborhoods R and R' of the critical points ω on B.

(a) *We choose R so small that any circuit on R is homologous to zero on B. See Lemma* 7.1. *We also choose R so as to be admissible in Theorem* 2.

(b) *We choose R' so small that any circuit on R' is homologous to zero on some region interior to R.*

(c) *If $k\neq 0$ we still further restrict R' so that R, R', and a positive constant e, taken less than $c-a$, satisfy Theorem* 2.

(d) *If $k=0$ we restrict R' so that on R', $J(\pi)>c$ except at the critical points ω and their limit points.*

Let C_j be any complex on B whose boundary C_{j-1} lies on A. An application of Lemma 7.3 gives the following homology and congruence:

$$C_j \sim C_j^1 + C_j^2 \quad \text{on } B, \tag{9.1}$$

$$C_{j-1} \equiv C_j^1 + C_j^2 \quad \text{on } B, \tag{9.2}$$

where C_j^1 is on R' and C_j^2 satisfies

$$J(\pi) < c - e_1, \quad e_1 < e, \tag{9.3}$$

where e_1 is any sufficiently small positive constant. We have taken $e_1 < e$. Suppose C_{j-1}^1 is the common boundary* of C_j^1 and C_j^2. We have then

$$C_j^1 \equiv C_{j-1}^1 \qquad \text{on } R', \tag{9.4}$$

$$C_j^2 \equiv C_{j-1}^1 + C_{j-1} \quad \text{on } B \text{ and } (9.3). \tag{9.5}$$

Lemma 9.1. *If $j \neq k$, C_j is homologous on B to some j-complex on A.*

We divide the proof into three cases.

Case 1. $j \neq 0$, $k \neq 0$. Because R, R', and e satisfy Theorem 2, and because $j \neq k$, there exists a complex C_j^3 such that [see Theorem 2, (a)]

$$C_{j-1}^1 \equiv C_j^3 \quad \text{on } R \text{ and } (9.3). \tag{9.6}$$

From (9.4) and (9.6) we see that $C_j^1 + C_j^3$ is without boundary and on R. Because R satisfies the restriction (a) we have then

$$C_j^1 + C_1^3 \sim 0 \quad \text{on } B. \tag{9.7}$$

From (9.1) and (9.7) we have

$$C_j \sim C_j^2 + C_j^3 \quad \text{on } B. \tag{9.8}$$

Now both C_j^2 and C_j^3 satisfy (9.3). With the aid of J-deformations we see that $C_j^2 + C_j^3$ is homologous to a complex K_j on A, with the same boundary on A. Hence

$$C_j \sim K_j \quad \text{on } B$$

and the proof is complete in Case 1.

Case 2. $j=0$, $k \neq 0$. For $j=0$, (9.1) still holds. We next define C_j^3 as any set of points on R and on (9.3) equal in number to the points of C_j^1. The proof may now be started with (9.7), and proceeds therefrom as before.

Case 3. $j \neq 0$, $k=0$. Here C_{j-1}^1 lies on R' and satisfies (9.3), which offers a contradiction to (d) unless C_{j-1}^1 is null. Thus C_{j-1}^1 is null. We now proceed exactly as in Case 1, understanding, however, that both members of (9.6) are now null.

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* We admit the possibility that C_{j-1} or C_{j-1}^1 be null. For $j=0$ we regard them as meaningless.

10. **Homologies among k-circuits when k is the type number ($k \neq 0$).** We understand that R and R' are chosen as in the preceding section. According to Theorem 2 there then exists a $(k-1)$-cycle D_{k-1} on R' and (9.3) which bounds no complex on R and (9.3). Since R' satisfies (b) of §9 we have, however,

$$D_{k-1} \equiv D_k^1 \quad \text{on } R, \tag{10.1}$$

where D_k^1 is a k-complex interior to R.

We now come to a division into two cases according to the nature of B.

Case α. There exists a complex D_k^2 such that

$$D_{k-1} \equiv D_k^2 \quad \textit{on } (9.3) \textit{ and } B. \tag{10.2}$$

Case β. No such complex as D_k^2 exists.

LEMMA 10.1. *In Case α, the k-cycle D_k defined as*

$$D_k^1 + D_k^2 = D_k \quad \textit{on } B \tag{10.3}$$

is not homologous on B to any cycle on A.*

Suppose the lemma false and that D_k were homologous on B to a k-cycle D_k^3 lying on A. We would have then

$$D_k^1 + D_k^2 + D_k^3 \equiv D_{k+1} \quad \text{on } B \tag{10.4}$$

where D_{k+1} is a complex on B.

We can further suppose that D_{k+1} consists of points which either satisfy (9.3), or else are interior to R. For as in §7 we could J-deform D_{k+1} into such a complex. Such a deformation, say T', would alter the boundary of D_{k+1}, but it could be replaced by a deformation which did not alter this boundary. One would need simply to hold the boundary fast during T' and hold points near this boundary fast during a portion of the deformation starting from a suitable time depending continuously on the distance to the boundary.

With this understood, let D_k^4 be the boundary of the sum of those $(k+1)$-cells of D_{k+1} which do not wholly satisfy (9.3), or else whose boundaries do not wholly satisfy (9.3). We see that D_k^4 is on (9.3) except for such k-cells as it has in common with D_k^1. If D_{k+1} has been sufficiently finely divided D_k^4 would also lie on R, as we now suppose. Thus

$$D_k^1 + D_k^4 \tag{10.5}$$

is on R, and if reduced, mod 2, satisfies (9.3). But since D_k^4 is without boundary, we have from (10.1)

* Including the null cycle.

$$D_{k-1} \equiv D_k^1 + D_k^4. \tag{10.6}$$

Thus D_{k-1} would bound the complex (10.5) which is on R and (9.3), contrary to the choice of D_{k-1}.

The lemma is thereby proved.

To proceed further it will be convenient to divide the complexes C_k of §9 ($k \neq 0$) into two classes according to the nature of the complex C_{k-1}^1 associated with C_k by (9.4) and (9.5). The distinctive properties of these classes are the following:

Class I. $C_{k-1}^1 \sim 0$ on R and (9.3).

Class II. $C_{k-1}^1 \sim D_{k-1}$ on R and (9.3).

That this division is exhaustive follows from Theorem 2.

LEMMA 10.2. *In Case α, each k-cycle C_k on B is homologous on B to a linear combination of the cycle D_k and cycles on A.*

C_k *in Class* I. Here there exists a complex C_k^3 such that

$$C_{k-1}^1 \equiv C_k^3 \quad \text{on } R \text{ and } (9.3). \tag{10.7}$$

According to (9.4) and (10.7), $C_k^1 + C_k^3$ is a k-cycle. Since it is on R we have

$$C_k^1 + C_k^3 \sim 0 \quad \text{on } B. \tag{10.8}$$

From (9.1) and (10.8) we have respectively

$$C_k \sim C_k^1 + C_k^2 \sim C_k^3 + C_k^2 \quad \text{on } B. \tag{10.9}$$

Now $C_k^3 + C_k^2$ satisfies (9.3), and so can be J-deformed on B into a k-cycle H_k on A. We thus have

$$C_k \sim H_k \quad \text{on } B.$$

Thus the lemma is proved when C_k is in Class I.

C_k *in Class* II. Here we have

$$C_{k-1}^1 + D_{k-1} \equiv C_k^4 \quad \text{on } R \text{ and } (9.3), \tag{10.10}$$

where C_k^4 is a complex on R and (9.3). Adding (9.4) for $k=j$, (10.1), and (10.10), we have

$$D_k^1 + C_k^1 + C_k^4 \equiv 0 \quad \text{on } R. \tag{10.11}$$

Now the complex (10.11) is a k-cycle on R so that

$$D_k^1 + C_k^1 + C_k^4 \sim 0 \quad \text{on } B. \tag{10.12}$$

From the definition of D_k and from (9.1) we have

$$D_k + C_k \sim D_k^1 + D_k^2 + C_k^1 + C_k^2 \quad \text{on } B. \tag{10.13}$$

Eliminating C_k^1 from (10.13) with the aid of (10.12) we have

$$D_k + C_k \sim D_k^2 + C_k^4 + C_k^2 \quad \text{on } B. \tag{10.14}$$

But the right member of (10.14) satisfies (9.3) and is accordingly homologous on B to a cycle H_k on A. With the aid of (10.14) we have then

$$C_k \sim D_k + H_k \quad \text{on } B,$$

and the lemma is proved when C_k is in Class II.

LEMMA 10.3. *In Case β each k-cycle C_k on B is homologous on B to a k-cycle on A.*

If C_k belongs to Class I the proof in the preceding lemma applies. We shall show that in Case β, Class II is empty.

If C_k belonged to Class II, (10.10) would hold as described. Adding (10.10) and (9.5) for $j=k$, and noting that $C_{k-1}=0$, we have

$$D_{k-1} \equiv C_k^2 + C_k^4 \quad \text{on } B \text{ and } (9.3),$$

contrary to the hypothesis distinguishing Case β.

11. **Homologies among $(k-1)$-circuits, $k\neq 0$.** In this section we prove three lemmas.

LEMMA 11. 1. *In Case α any cycle C_{k-1} on A which is homologous to zero on B will be homologous to zero on A.*

If C_{k-1} is homologous to zero on B it bounds a complex C_k which we identify with the complex C_j of §9. We turn next to C_{k-1}^1 defined by (9.4) and (9.5). According to Theorem 2, C_{k-1}^1 is either homologous to zero or to D_{k-1}, on R and (9.3). If C_{k-1}^1 is homologus to D_{k-1} we note that, in Case α, D_{k-1} is homologous to zero on B and (9.3). It is always true then in Case α that there exists a k-complex C_k^5 such that

$$C_{k-1}^1 \equiv C_k^5 \quad \text{on } B \text{ and } (9.3). \tag{11.1}$$

Upon adding (9.2), (9.4) and (11.1) we have

$$C_{k-1} \equiv C_k^2 + C_k^5 \quad \text{on } B \text{ and } (9.3). \tag{11.2}$$

Since $C_k^2+C_k^5$ satisfies (9.3) it can be J-deformed into a complex K_k on A without altering C_{k-1}. Thus C_{k-1} bounds K_k on A and the lemma is proved.

Definition of D_{k-1}^1. Denote by D_{k-1}^1 a $(k-1)$-cycle obtained by deforming D_{k-1} on B and (9.3) into a $(k-1)$-circuit on A. Such a deformation is possible since D_{k-1}^1 satisfies (9.3). According to the definition of D_{k-1}^1 we have

$$D_{k-1}^1 + D_{k-1} \equiv D_k^5 \quad \text{on } B \text{ and } (9.3), \tag{11.3}$$

where D_k^5 is a k-complex on B and (9.3). We note that

$$(11.4) \qquad D_{k-1}^1 \sim 0 \quad \text{on } B.$$

For upon adding (11.3) and (10.1) we have

$$D_{k-1}^1 \equiv D_k^1 + D_k^5 \quad \text{on } B.$$

Lemma 11.2. *In Case α, D_{k-1}^1 is homologous to zero on A, but in Case β, D_{k-1}^1 is not homologous to zero on A.*

That D_{k-1}^1 is homologous to zero on A in Case α follows from (11.4) and Lemma 11.1.

If in Case β we had a congruence

$$(11.5) \qquad D_{k-1}^1 \equiv D_k^6,$$

where D_k^6 is a k-complex on A, then upon adding (11.3) and (11.5) we would have

$$D_{k-1} \equiv D_k^5 + D_k^6.$$

But $D_k^5+D_k^6$ satisfies (9.3) contrary to the nature of D_{k-1} in Case β, and the lemma is proved.

Lemma 11.3. *In Case β, any cycle C_{k-1} on A which is homologous to zero on B is either homologous to zero or to D_{k-1}^1 on A.*

If C_{k-1} is homologous to zero on B it bounds a complex C_k which can be identified with the complex C_j of §9.

If C_k is of Class I we repeat the reasoning of Lemma 10.2 under Class I except that we here follow (10.9) with the statement that $C_k^3+C_k^2$ can be J-deformed on B into a complex K_k on A without altering C_{k-1}. Thus C_{k-1} bounds K_k on A and the lemma is proved if C_k is in Class I.

If C_k is in Class II, (10.10) holds as before. Upon adding (9.2) and (9.4), for $j=k$, to (10.10) and (11.3), we have

$$(11.6) \qquad D_{k-1}^1 + C_{k-1} \equiv D_k^5 + C_k^4 + C_k^2.$$

But the right member of (11.6) satisfies (9.3), from which fact we infer that

$$D_{k-1}^1 \sim C_{k-1} \quad \text{on } A$$

and the proof is complete.

12. The changes in the connectivities when $k\neq 0$. In this section we prove two theorems.

THEOREM 3. *Let a and b be two non-critical values of $J(\pi)$, $a<b$, between which J takes on a critical value corresponding to just one extremal of type k. The connectivities R_j of the domains $A=(a, \rho)$ and $B=(b, \rho)$ differ at most when $j=k$ or $k-1$.*

If $(K)_j$ be a complete j-set for A, it follows from Lemma 9.1 for $j\neq k$ that any j-cycle on B is homologous on B to a linear combination of members of $(K)_j$. From the same lemma it follows for $j\neq k-1$ that there can be no proper homologies on B among the members of $(K)_j$ without there being proper homologies among the members of $(K)_j$ on A. Thus $(K)_j$ is a complete j-set for B and the theorem is proved.

THEOREM 4. *If $k\neq 0$ the connectivities R_j of the domains A and B differ only as follows:*

$$\text{Case } \alpha: \qquad \Delta R_k = 1; \tag{12.1}$$

$$\text{Case } \beta: \qquad \Delta R_{k-1} = -1, \tag{12.2}$$

where Δ refers to the change as we pass from A to B.

We shall first treat Case α.

Let $(K)_k$ be a complete k-set for A. I say that $(K)_k$ together with D_k of §10 will form a complete k-set for B.

For it follows first from Lemma 10.2 that every k-cycle on B is homologous on B to a linear combination of D_k and the members of $(K)_k$.

Secondly there can be on B no proper homologies involving the members of $(K)_k$ and D_k. For if a combination of members of $(K)_k$ alone were bounding on B, with the aid of Lemma 9.1 we could infer that the same combination would be bounding on A, contrary to the nature of $(K)_k$. But no linear combination of D_k and the members of $(K)_k$ which actually involved D_k could be homologous to zero on B without contradicting Lemma 10.1.

Thus $(K)_k$ with D_k forms a complete k-set for B so that $\Delta R_k=1$.

We must now show that $\Delta R_j=0$ when $j\neq k$. This is already established in Theorem 3 provided $j\neq k-1$. When $j=k-1$, Lemmas 9.1 and 11.1 show that a complete j-set for A is a complete j-set for B. The theorem is thus established for Case α.

We can now treat Case β.

Turning to Lemma 10.3 and Theorem 3 we see readily that $\Delta R_j=0$ for $j\neq k-1$.

We shall show finally that (12.2) holds.

According to Lemma 11.2, D_{k-1}^1 is not homologous to zero on A, and so there exists a complete $(k-1)$-set for A, say $(K)_{k-1}$, of which D_{k-1}^1 is a member. Let $(H)_{k-1}$ be the set of $(k-1)$-circuits in this set after D_{k-1}^1 has been removed. I say that $(H)_{k-1}$ is a complete $(k-1)$-set for B.

For according to Lemma 9.1 and the homology (11.4) any $(k-1)$-cycle on B is homologous on B to a combination of $(k-1)$-cycles from $(H)_{k-1}$. That there are no homologies on B among members of $(H)_{k-1}$ follows readily from Lemma 11.3. Thus (12.2) holds as required and the theorem is proved.

13. **The case of minima, $k=0$.** Here the value $J(\pi)=c$ furnishes a relative minimum to $J(\pi)$ on B. We understand a relative minimum to include an absolute minimum as a special case.

Let P be a point of the critical set ω.

LEMMA 13.1. *If c is a relative minimum for $J(\pi)$ on B, then any 0-cycle C_0 on B is homologous on B to a 0-cycle on A, or a combination of 0-cycles on A and the 0-cycle P.*

According to Lemma 7.3 we have the homology

$$C_0 \sim C_0^1 + C_0^2 \quad \text{on } B, \tag{13.1}$$

where C_0^1 is a set of r points on R', and C_0^2, possibly null, is a set of points on B at which $J(\pi)<c$. Note first that C_0^2 is homologous on B to a set K_0 of points on A so that

$$C_0 \sim C_0^1 + K_0 \quad \text{on } B. \tag{13.2}$$

We distinguish between the cases r even and r odd.

If r is even, C_0^1, according to the choice of R', is homologous to zero on B, so that C_0 is homologous to K_0 on B and the lemma is proved. If r is odd, we have

$$C_0 \sim P + K_0 \quad \text{on } B,$$

and the lemma is proved for all cases.

LEMMA 13.2. *If $J(\pi)=c$ is a relative minimum for $J(\pi)$ on B, the point P on ω cannot be connected on B with any point on A.*

To prove this, note that at the boundary points of R' which are not also boundary points of B, $J(\pi)>c+e$, where e is a positive constant.

Suppose P could be joined to a point on A by a continuous curve C_1 on B. A sufficient number of iterations of the deformation T would carry C_1 into a curve C_1^1 on B on which $J(\pi)<c+e$. But it would also carry P into a point on ω in R', so that C_1^1 would still join a point on ω in R' to a point on A. This is impossible, since on C_1^1, $J(\pi)<c+e$. Thus the lemma is proved.

If $J(\pi)=c$ gives an absolute minimum to $J(\pi)$ on B, A will be devoid of points. In that case we say that the *connectivities of A are all zero*. With this understood we have the following theorem.

THEOREM 5. *If $k=0$ the connectivities R_i of the domains A and B differ only in that $\Delta R_0=1$, where ΔR_0 is the change in R_0 as we pass from A to B.*

That R_0 is the only connectivity to change as we pass from A to B follows from Theorem 3.

Now if $(K)_0$ is a complete 0-set for A, null if A is null, then $(K)_0$ with P will be a complete 0-set $(H)_0$ for B.

That all 0-cycles on B are homologous on B to members of $(H)_0$ is affirmed by Lemma 13.1.

That there are no homologies on B among members of $(H)_0$ can be seen as follows. There are no homologies on B among members of $(K)_0$. For such homologies would lead, upon applying Lemma 9.1 for $j=1$, to the conclusion that there were homologies on A among the same members of $(K)_0$. Any homology on B among the members of $(H)_0$ must then explicitly involve P which would mean that P on ω would be connected on B to some point on A contrary to Lemma 13.2. Thus there are no homologies on B among the members of $(H)_0$.

Thus $(K)_0$ and P form a complete 0-set for B. The theorem follows at once.

III. THE EXISTENCE OF EXTREMALS AND THEIR RELATIONS

14. Relations between the extremals and the connectivities. *It will be convenient to call the extremal appearing in Theorem 4 of increasing or decreasing type according as $\Delta R_k=1$ or $\Delta R_{k-1}=-1$.*

In accordance with Theorem 5 each extremal of type zero should be classified as increasing in type.

We come now to the following principal theorems. As previously we suppose a is a non-critical value of J, and that $P\neq Q$.

THEOREM 6. *If among the extremals g on which $J<a$ there are none on which P is conjugate to Q, then between the integers M_k which give the total number of these extremals of type k, and the connectivities R_i of the domain (a,ρ), the following relations hold:*

$$\begin{gathered} M_0 \geqq R_0, \\ M_0 - M_1 \leqq R_0 - R_1, \\ M_0 - M_1 + M_2 \geqq R_0 - R_1 + R_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ M_0 - M_1 + \cdots + (-1)^r M_r = R_0 - R_1 + \cdots + (-1)^r R_r, \end{gathered} \tag{14.1}$$

where r is the maximum of the type numbers k.

Consider first the case where the J-lengths of the different extremals g are all different.

Let the extremals of type i be further divided into p_i extremals of increasing type, and q_i extremals of decreasing type. We have at once

$$M_i = p_i + q_i, \tag{14.2}$$

$$R_i = p_i - q_{i+1} \qquad (i = 0, 1, \cdots, r), \tag{14.3}$$

where, in particular, q_0 and q_{r+1} equal zero. From the relations (14.2) and (14.3) we may eliminate p_i and find that

$$M_0 - M_1 + \cdots + (-1)^i M_i = R_0 - R_1 + \cdots + (-1)^i R_i + (-1)^i q_{i+1}. \tag{14.4}$$

Relations (14.1) follow at once from (14.4).

If now two or more of the extremals joining P to Q have the same J-length it follows from Lemma 16.2 that Q can be replaced by a point Q', arbitrarily near Q, and such that for Q' the J-lengths of the different extremals g are all different.

For the pair PQ' the relations (14.1) now hold. But as Q' approaches and takes the position Q, the numbers M_k will not change, since P is not conjugate to Q on any of the extremals g. Neither will the connectivities R_i change, as follows from Theorem 13 of §18.

Thus the relations (14.1) must hold in all cases and the theorem is proved.

We have the following corollary.

Corollary. *If the ith connectivity of (a, ρ) is R_i there exist at least R_i extremals joining P to Q of type i.*

To proceed further it will be convenient to introduce the conception of the *conjugate sequence* for the pair PQ.

Let P and Q be any pair of points on S. Let N_k be the number of closed extremal segments joining P to Q on which there are k points conjugate to P, counting conjugate points according to their orders. If there are an infinity of such extremals we replace N_k by ∞.

The sequence

$$N_0 N_1 N_2 \cdots \tag{14.5}$$

will be called the conjugate sequence for P and Q.

We shall investigate this sequence first for pairs P and Q which are non-specialized in the following sense.

Two distinct points P and Q which are not conjugate to each other on any extremals will be termed non-specialized.

In §16 we shall prove the following.

There exist non-specialized pairs of points PQ in the respective neighborhoods of any two points on S.

We shall begin by supposing S restricted to the most important particular case.

15. **The case where the region S is elementary.** We suppose now that S is an elementary region in the sense that it is homeomorphic with an m-sphere and its interior.

For the moment let us term a curve on s *admissible* if it has a continuously turning tangent except at most at a finite number of points, and joins P to Q. The fact that S is elementary as well as extremal-convex leads to the following lemma.

LEMMA 15.1. *On S there exists a continuous deformation δ that deforms all admissible curves for which $J<a$ through admissible curves of J-lengths less than some constant b, into a single admissible curve, thereby deforming continuous families of such curves continuously.*

Since S is elementary there exists a continuous deformation, say δ_1, of S on itself, leaving P and Q fixed, that carries S into a set of points on a curve γ joining P to Q. We can moreover suppose, for simplicity, that γ is a minimizing extremal joining P to Q.

Now let each admissible curve g for which $J<a$ be divided into $r+1$ successive segments g_i of equal J-length. Let us suppose r taken so large that the end points of g_i and their images under δ_1 can be joined by elementary extremals.

We now define the deformation δ. On each curve g let each segment g_i be replaced by the elementary extremal g_i' that joins its end points, or more particularly let a point on g_i dividing g_i in a certain ratio be replaced by that point on g_i' which divides g_i' in the same ratio. Each curve g can be readily deformed through admissible curves into the corresponding curve g' by a deformation that deforms continuous families of such curves continuously.* To deform further the resultant curves g' into a single admissible curve we require the end points of g_i' to move according to the deformation δ_1, requiring a point on g_i' which divides g_i' in a certain ratio to be deformed into that point of the moving elementary extremal g_i'' which divides g_i'' in the same ratio.

Each curve g will thereby be deformed into a broken extremal g'' with its vertices on γ. Finally let these vertices move along γ each at a constant J-rate into a set of vertices which divide γ into a set of elementary extremals

* See the deformation D'; Morse II p. 263.

of equal J-length, thereby deforming g'' into a single admissible curve γ in the desired manner.

If the elementary extremals used have J-lengths at most ρ, the curves used in the deformation will have J-lengths at most $r\rho$. Thus b can be any constant $b > \rho r$.

LEMMA 15.2. *The constant b of Lemma* 15.1 *can be chosen independently of the position of P and Q on S.*

To prove this let P' and Q' be any other pair of points on S and let g be an admissible curve joining them. Let p^+ and q^+ be minimizing extremals joining PP' and QQ' respectively in the senses indicated. Holding P' and Q' fast let g be deformed into the curve g_1 consisting of the sequence of curves $(p^-p^+gq^-q^+)$ joining P' to Q'. Note that this deformation can be made without increasing J-lengths by more than $4d$, where d is the maximum of the J-lengths of minimizing extremals joining any two points of S.

Now the curve (p^+gq^-) gives a subsegment of g_1 joining P to Q. The class of such subsegments can be deformed through admissible curves joining P to Q into a single curve γ joining P to Q, using thereby curves all of J-lengths less than some constant b_1, as is affirmed by the preceding lemma.

The class of curves g will thereby be deformed into the curve $(p^-\gamma q^+)$ through the mediation of admissible curves joining P' to Q' all of J-lengths less than b_1+4d. The latter constant can be taken as our choice of b.

Thus the statement in italics is proved.

LEMMA 15.3. *All circuits on the domain (a, ρ) are homologous to zero on a domain (b, ρ) where b is any constant sufficiently large chosen independently of P and Q.*

It will be sufficient to choose b as any constant at least as great as the constant b of Lemma 15.2. As previously we suppose J_0 greater than a and b, and n chosen as in §3.

Let C_i be any i-circuit on (a, ρ). If the broken extremals corresponding to C_i be subjected to the deformation δ of the preceding lemmas they will be carried into the curve γ. We can set up a corresponding deformation of the points (π) of C_i as follows.

We first apply the deformation D of §5 to C_i. The end points of the resulting elementary extremals will divide each original curve g defined by by C_i into $n+1$ segments of equal J-lengths. Let these end points now take those positions on the variable curve g' replacing g during δ which divide g' into segments of equal J-length. The resulting deformation will carry C_i into a point $(\pi)_0$ corresponding to γ.

The lemma follows directly.

We come now to the following theorem.

THEOREM 7. *Corresponding to the set of extremals which join two non-specialized points P and Q on which $J<a$, there always exists a complementary set of extremals on which $a<J<b$ such that for the combined sets*

$$
\begin{aligned}
m_0 &\geqq 1,\\
m_0 - m_1 &\leqq 1,\\
\cdots\cdots&\cdots\cdots\\
m_0 - m_1 + \cdots + (-1)^r m_r &= 1 + (-1)^r R_r,
\end{aligned}
\tag{15.1}
$$

where m_i is the number of extremals of type i in the combined sets and r is the maximum of the type numbers of the extremals of the original set and where R_r is the rth connectivity of (a, ρ).

As previously we need prove the theorem only for the case where the J-lengths of the different extremals joining P to Q are different.

The relations (14.1) of Theorem 6 furnish our starting point.

From Lemma 15.3 we see that all circuits on (a, ρ) are homologous to zero on a domain (b, ρ) for which b is large enough.

If R_i is the ith connectivity of (a, ρ) we see that there must be at least R_i extremals for $i>0$, and R_0-1 for $i=0$, of decreasing type $i+1$ with $a<J<b$. We take these extremals $(i=0, \cdots, r)$ as the complementary set. In terms of the integers M_i and R_i of Theorem 6 we have

$$
m_0 = M_0, \quad m_1 = M_1 + (R_0 - 1),
$$
$$
m_i = M_i + R_{i-1} \qquad (i = 1, \cdots, r).
$$

Relations (15.1) follow now with the aid of (14.1).

We note that, in Theorem 7, b is a sufficiently large positive constant which may, in particular, be chosen independently of the position of P and Q on S.

We turn now to the conjugate sequences (14.5).

THEOREM 8. *If none of the integers N_i of the conjugate sequence of a non-specialized pair of points are infinite, they satisfy the infinite set of inequalities*

$$
\begin{aligned}
N_0 &\geqq 1,\\
N_0 - N_1 &\leqq 1,\\
N_0 - N_1 + N_2 &\geqq 1,\\
\cdots&\cdots
\end{aligned}
\tag{15.2}
$$

If all of the integers of the conjugate sequence are finite up to N_{k+1}, the first k $+1$ relations in (15.2) *still hold.*

The first statement in the theorem is a consequence of the last. Let us turn then to the last.

Suppose all of the integers up to N_{k+1} are finite. Let a be a non-critical value of J greater than the J-lengths of all extremals with types at most k. If now we apply Theorem 7 we find that we must have

$$m_i = N_i \qquad (i = 0, 1, \cdots, k)$$

and the first $k+1$ relations in (15.2) follow from (15.1).

The theorem follows directly.

COROLLARY. *If there are no extremals joining P and Q upon which there are k conjugate points, then there are either an infinite number of extremals upon which there are fewer than k conjugate points, or else the numbers $N_0, \cdots, N_{k-1}$ satisfy*

$$N_0 - N_1 + \cdots + (-1)^{k-1} N_{k-1} = 1.$$

This follows from the kth and $(k+1)$st inequalities of (15.2).

We come now to a theorem in which we shall not require the numbers N_i to be finite.

If N_r represents ∞ we shall understand $N_i - 1$ and $N_i + 1$ as also representing ∞.

THEOREM 9. *Let P and Q be any two non-specialized points on S.*

(a) *If there are N_k extremals of type $k>1$ there are at least N_k extremals of the two adjacent types.*

(b) *If there are N_1 extremals of type one there are at least $N_1 - 1$ extremals of the two adjacent types.*

(c) *If there are N_0 extremals of type zero there are at least** $N_0 - 1$ *extremals of the adjacent type one.*

We shall first prove (a).

From the first inequality in (15.1) which involves m_{i+1}, and the third preceding inequality, we find that

$$m_i \leqq m_{i-1} + m_{i+1}, \quad i > 1.$$

If N_{i-1}, N_i, and N_{i+1} are finite, and we take the constant a in Theorem 7 large enough, these N's become equal to m_{i-1}, m_i and m_{i+1} respectively, and (a) is proved for this case.

* Part (c) is the representation here of the "minimax principle" of Birkhoff. See *Dynamical systems with two degrees of freedom*, these Transactions, vol. 18 (1917), p. 249.

If N_i is infinite there must be extremals of type i of arbitrarily great J-length. If we take a successively as the constants of a sequence of constants becoming infinite, m_i will become infinite, and hence from (15.1) either m_{i-1} or else m_{i+1}. Part (a) then follows in this case as well.

Parts (c) and (b) follow similarly from the second and third inequalities in (15.1).

IV. The density of conjugate points, and invariance of the connectivities

16. **Specialized points P and Q.** We wish to show that the so called non-specialized pairs of points are really general. We begin with the following lemma.

LEMMA 16.1. *The conjugate points of a point P at distances from P along the corresponding extremals not exceeding a positive constant d, form a set which is nowhere dense.*

Let (u) represent a point in an auxiliary m-space. Let each extremal issuing from P be represented in the space (u) by a ray issuing from the origin with a direction parallel to its direction at P, and with points on the extremal at distances s from P, corresponding to points on the ray at distances s from the origin. The corresponding functions

$$x_i = x_i(u) \qquad (i = 1, 2, \cdots, m)$$

will be analytic except at the origin. The jacobian $D(u)$ of these functions will vanish at the conjugate points. Its rank r at such points will be between 0 and m (Morse III §7).

Suppose $(x)_0$ is a conjugate point corresponding to a point $(u)_0$. If $D(u_0)$ is of rank r one sees that there is an r-plane X through $(x)_0$ in the space (x), such that the distance of the points $[x(u)]$ from X is an infinitesimal of at least the second order with respect to the distance ρ of (u) from $(u)_0$.

Let S_1 be the interior of an $(m-1)$-sphere of radius ρ with center at $(u)_0$. One sees that the points (x) corresponding to points (u) on S_1 can be enclosed in a volume V whose ratio to that of S_1 will approach zero as ρ approaches zero. For V one could take a generalized cylinder consisting of the points P of X at a distance $c\rho$ from $(x)_0$ together with the points on perpendiculars to X, points at a distance $h\rho^2$ from these points P, where c and h are suitably chosen positive constants independent of the choice of $(u)_0$.

Let e now be an arbitrarily small positive constant. Let us break the space (u) up into congruent m-cubes. If the diameter of each of these m-cubes be sufficiently small, then such of the corresponding sets $[x(u)]$ as contain conjugate points with $s \leqq d$ can be enclosed in elementary volumes such as V whose ratios to that of the cubes will be less than e. The sum of these volumes V will be less than e times the total volume of the corresponding cubes. The sum of these elementary volumes will then be arbitrarily small. This is possible only if the conjugate points in question are nowhere dense.

THEOREM 10. *The set of all conjugate points of a fixed point P is nowhere dense on S.*

Let $d_1, d_2, \cdots$ be an increasing set of positive constants which become infinite with their subscripts. Let Q be any point on S. It follows from the preceding lemma that there exists in any neighborhood of Q a sequence of $(m-1)$-spheres

$$S_1, S_2, \cdots$$

each within the other, and such that there are within S_i no points conjugate to P for which $s < d_i$. These $(m-1)$-spheres will have at least one common interior point, say A. The point A cannot be a conjugate point of P without violating the principle under which the $(m-1)$-spheres S_i were stated to exist.

Thus the theorem is proved.

This theorem amounts to the statement already made that there exist non-specialized pairs of points P and Q in the respective neighborhoods of any two given points of S.

The following theorem is a strong aid in proving the existence of extremals joining specialized pairs of points.

THEOREM 11. *If each pair of points in a set of non-specialized points can be joined by an extremal γ which is bounded in J-length for the set, and of type k, then any limit pair P_0Q_0, $P_0 \neq Q_0$, of pairs of the set can be joined by an extremal g on which there are at least k, and at most $k+m-1$, points* conjugate to P_0.*

Let PQ represent any pair of points in the set. As PQ approaches P_0Q_0, the initial directions of the extremals γ will have at least one limit direction. Let g be the extremal with this limit direction.

The extremal g will have at least k conjugate points on it. For otherwise there would be fewer than k conjugate points on extremals γ neighboring g. See Morse IV §9.

Suppose there were $k+r+s$ conjugate points on g, where s is the number of conjugate points to be counted at Q_0. Now $r \leqq 0$, for otherwise extremals γ neighboring g would have at least $k+r>k$ conjugate points on them. Finally

* Counting conjugate points according to their orders.

$s \leqq m-1$ as was shown in Morse III §7. Thus there are at most $k+m-1$ conjugate points on g.

The following two lemmas have already been used. They can be conveniently proved here.

LEMMA 16.2. *Let PQ be a pair of distinct points, and d any positive constant. There exists in the neighborhood of the point Q at least one point which is joined to P by no two extremals whose J-lengths are equal and less than d.*

According to Lemma 16.1 there exists in the neighborhood of Q at least one point Q' which is joined to P by no extremals on which $J \leqq d$ and on which Q' is conjugate to P. Suppose, however, that there are at least two extremals g and g' joining P to Q' with equal J-lengths. Suppose their direction cosines at Q' are (p) and (q).

Since Q' is not conjugate to P, a slight variation of Q' will cause a slight variation of the extremals from P to Q' neighboring the initial g and g'. The J-lengths of these extremals will be analytic functions of the coördinates of Q'.

The difference of the J-lengths of the extremals g and g' will have partial derivatives given by (4.1), and as seen in §4, not all of these partial derivatives are zero for $(p) \neq (q)$. The locus of points Q' neighboring the initial position of Q' for which two or more of the extremals have J-lengths which are equal and less than d or near d, will thus lie on a finite number of analytic $(m-1)$-dimensional manifolds without singularities. The lemma follows at once.

With the aid of Lemmas 16.1 and 16.2 one proves the following lemma. The proof is similar to that of Theorem 10.

LEMMA 16.3. *For a fixed point P there exists in the neighborhood of every point $Q \neq P$ at least one point which is not a conjugate point of P nor which is joined to P by more than one extremal of any one J-length.*

17. **Example. Geodesics on a knob.** Suppose we have an analytic surface S, without singularities, with boundary B, and homeomorphic to a circular disc. On S consider the integral of arc length. Suppose S is extremal-convex. Suppose S possesses a knob-shaped protuberance. More exactly suppose there is a portion of S homeomorphic to a circular disc, and bounded by a closed geodesic g that is shorter than nearby closed curves.

Let R be the region between g and B. Let P and Q be any pair of nonspecialized points on R. The geodesics γ which give an absolute minimum to the arc length relative to all other admissible curves on R which join P to Q, and are deformable into γ on R, are infinite in number and of type zero. From Theorem 9 we have the following.

If P and Q are any two non-specialized points on R there will be an infinite set of geodesics of type one joining P to Q on S.

By the use of Theorems 7 and 11 it is not difficult to prove the following more general theorem.

If P and Q are any two points whatsoever on R, there will be an infinite set of geodesics joining P to Q, upon each of which there will be at least one point conjugate to P.

18. **The invariance of the connectivities.** We shall first concern ourselves with the dependence of the connectivities of a domain (a, ρ) upon the choice of n, the number of vertices in (π). It will be convenient to indicate the apparent dependence of (a, ρ) on n by now representing this domain by (a, ρ, n). We shall prove the following.

THEOREM 12. *The connectivity numbers are independent of n, in the sense that the connectivity numbers of (a, ρ, n) equal those of (a, ρ, n'), where n and n' are any two admissible choices of n.*

Let there be given a point $(\pi)'$ on (a, ρ, n'), determining a broken extremal g'. Let $a(\pi')$ stand for the point (π) whose n vertices divide g' into $n+1$ successive segments h_i of equal J-length. Let g be the admissible broken extremal determined by (π). We shall prove the following statement.

(a) *The curve g' can be continuously deformed on S into the curve g through the mediation of ordinary curves which join P to Q, and whose J-lengths do not exceed that of g'.*

We shall take the time t as the parameter of our deformation, and let it vary from 0 to 1.

Suppose the end points of the segment h_i of g' are joined by an elementary extremal k_i of g. For each value of t we suppose h_i divided into two successive segments the ratio of whose J-lengths is that of t to $1-t$. For each value of t from 0 to 1 we now replace the second of these segments of h_i by itself, while we replace the first by an elementary extremal that joins its end points. In this manner h_i will be deformed into k_i, and thereby g' into g. Thus the statement (a) is proved.

Each point (π) determines a point $b(\pi)=(\pi)'$ exactly as (π) determines $a(\pi')=(\pi)$. Statement (b) will now be proved.

(b) *For every point (π) on the domain (a, ρ, n) the point $(\pi)''=a(b(\pi))$ lies on the domain (a, ρ, n) also. Moreover, there exists a deformation F on (a, ρ, n), of the points (π) on (a, ρ, n), which carries these points into the corresponding points $(\pi)''$.*

Let g, g' and g'' be respectively the broken extremals determined by (π), $b(\pi)$ and $(\pi)''$. By deformations similar to those described under (a),

g can be deformed into g', and thence into g'' without increasing J. Let t be a parameter of the resultant deformation, and vary from 0 to 1. Let $Z(t)$ be the curve that thereby replaces g at the time t. A point U on $Z(t)$ may be *determined* by giving t and the J-coördinate u of the point U on $Z(t)$. See §5.

A deformation F of the point (π) into the corresponding point $(\pi)''$ will now be defined. In the space of the points (x) let each vertex of (π) move to the corresponding vertex of $(\pi)''$ in such a manner that the pair (t, u) which *determines* (see above) this variable vertex moves on a straight line in the (t, u) plane at a constant velocity equal to the distance to be traversed. The corresponding deformation F is readily seen to have the desired properties.

If C_j is any complex on the domain (a, ρ, n) we shall denote by $b(C_j)$ that complex on (a, ρ, n') which consists of the images $b(\pi)$ of points (π) on C_j. Reciprocally if C_j' is a complex on (a, ρ, n'), $a(C_j')$ will denote the set of images $a(\pi')$ of points $(\pi)'$ on C_j'. With this understood we now prove a final statement (c).

(c) *If the cycles*

$$C_j^1, \cdots, C_j^r \tag{18.1}$$

form a complete j-set for (a, ρ, n'), the cycles

$$a(C_j^1), \cdots, a(C_j^r) \tag{18.2}$$

will form a complete j-set for the domain (a, ρ, n).

Let C_j be any j-cycle on the domain (a, ρ, n). Then $b(C_j)$ will be a j-cycle on the domain (a, ρ, n'). Because (18.1) gives a complete j-set for (a, ρ, n') we have

$$b(C_j) + \Sigma C_j^i \equiv C_{j+1} \quad \text{on } (a, \rho, n'), \tag{18.3}$$

where Σ stands for a suitable sum of cycles (18.1) and C_{j+1} is a complex on (a, ρ, n'). From (18.3) we see that

$$a(b(C_j)) + \Sigma a(C_j^i) \equiv a(C_{j+1}) \quad \text{on } (a, \rho, n). \tag{18.4}$$

From (b) we see however that

$$a(b(C_j)) \sim C_j \quad \text{on } (a, \rho, n). \tag{18.5}$$

From (18.4) and (18.5) we see finally that

$$C_j \sim \Sigma a(C_j^i) \quad \text{on } (a, \rho, n). \tag{18.6}$$

Thus the jth connectivity number of (a, ρ, n) will be at most r. We can reverse the rôles of (a, ρ, n) and (a, ρ, n'). We infer then that the connectivities of (a, ρ, n) and (a, ρ, n') are the same. Thus (c) and the theorem are proved.

We now give another theorem on the invariance of the connectivities.

THEOREM 13. *The connectivities of the domain (a, ρ) remain unchanged during any continuous variation of the end points P and Q on S $(P \neq Q)$, provided the constant a remain a non-critical value of J.*

Let the domain (a, ρ) of admissible points (π), set up for a pair of points, PQ, now be indicated by

$$(a, \rho, P, Q). \tag{18.7}$$

If e be a sufficiently small positive constant the domain (18.7) can be J-deformed onto the domain (see § 6)

$$(a - e, \rho - e, P, Q). \tag{18.8}$$

If e_1 now be chosen as a positive constant less than e, and $P'Q'$ be a pair of points sufficiently near PQ, the domain

$$(a - e_1, \rho - e, P', Q') \tag{18.9}$$

will be included in the domain (18.7) and will include the domain (18.8).

Now the domain (18.7) can be J-deformed on itself into a set of points on (18.8), and hence into a set of points on (18.9). It follows that the connectivities of (18.7) equal those of (18.9).

This remains true for any smaller choice of e_1 and choice of $P'Q'$ sufficiently near PQ. But if e_1 be sufficiently small, and $P'Q'$ so near PQ that a remains a non-critical value of J, the domain

$$(a, \rho, P', Q') \tag{18.10}$$

can be J-deformed on itself into a set of points on (18.9) and hence has the connectivities of (18.9). Hence (18.10) and (18.7) have the same connectivities, and the theorem is proved.

19. Extremals on closed, regular, analytic manifolds. All of the previous developments go through for this case as in the case of the region S, except for obvious changes of which we will enumerate the most important.

All references to the boundary, or the extremal-convex hypothesis are to be omitted.

Instead of having one set of coördinates (x), it will in general be necessary to give the manifolds by a finite set of overlapping parametric representations.

All the extremals through a point P may go through a second point Q.

This case should be treated separately. It can never occur for a non-specialized pair of points.

The theory of extremals on elementary regions will apply to elementary regions on the manifold. The relations between the extremals and the connectivities of the domain (a, ρ) are given by Theorem 6.

The most important case is the case where the manifold is homeomorphic with an m-sphere. The following theorem will be proved in a later paper.

THEOREM 14. *On a manifold homeomorphic with an m-sphere there are infinitely many extremals joining any two fixed points, including extremals of arbitrarily great length, with arbitrarily many conjugate points on them.*

In this theorem it is understood that two extremals are counted as different if they have different lengths, even if they overlap. The proof of this theorem depends upon the preceding work together with a method of determining the connectivities R_i, called *the topological continuation of extremals*.

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